## Lecture 1: The Moore–Penrose inverse

A **generalized inverse** (**G.I.**) of an arbitrary matrix $A \in \mathbb{C}^{m\times n}$ is a matrix $X \in \mathbb{C}^{n\times m}$ that satisfies certain useful properties of an inverse, and reduces to it if $A$ is nonsingular. Several G.I.'s will be mentioned, and the most important one, the **Moore–Penrose inverse**, will be studied in detail. It is characterized as the unique solution $X$ of the 4 Penrose equations

$$(1)\ AXA = A;\quad (2)\ XAX = X;\quad (3)\ (AX)^* = AX;\quad (4)\ (XA)^* = XA,$$

or equivalently, as the unique solution of

$$AX = P_{R(A)},\ XA = P_{R(A^*)},$$

where $P_L$ is the orthogonal projector onto the subspace $L$.

(a) Existence and uniqueness.

(b) Properties.

(c) Connection to the Singular Value Decomposition.

(d) The volume of a matrix.

(e) Computations.

**References**

[1] R. Penrose, A generalized inverse for matrices, *Proceedings of the Cambridge Philosophical Society* **51**(1955), 406-413.

[2] A. B–I; T.N.E. Greville, *Generalized Inverses*, Springer–Verlag, 2003. ISBN 0-387-00293-6.

[3] A.B–I, A volume associated with $m \times n$ matrices, *Lin. Algeb. and its Appl.* **167**(1992), 87–111