

Lecture 1: The Moore–Penrose inverse

A **generalized inverse (G.I.)** of an arbitrary matrix $A \in \mathbb{C}^{m \times n}$ is a matrix $X \in \mathbb{C}^{n \times m}$ that satisfies certain useful properties of an inverse, and reduces to it if A is nonsingular. Several G.I.'s will be mentioned, and the most important one, the **Moore–Penrose inverse**, will be studied in detail. It is characterized as the unique solution X of the 4 Penrose equations

$$(1) AXA = A; \quad (2) XAX = X; \quad (3) (AX)^* = AX; \quad (4) (XA)^* = XA,$$

or equivalently, as the unique solution of

$$AX = P_{R(A)}, \quad XA = P_{R(A^*)},$$

where P_L is the orthogonal projector onto the subspace L .

- (a) Existence and uniqueness.
- (b) Properties.
- (c) Connection to the Singular Value Decomposition.
- (d) The volume of a matrix.
- (e) Computations.

References

- [1] R. Penrose, A generalized inverse for matrices, *Proceedings of the Cambridge Philosophical Society* **51**(1955), 406-413.
- [2] A. B-I; T.N.E. Greville, *Generalized Inverses*, Springer-Verlag, 2003. ISBN 0-387-00293-6.
- [3] A.B-I, A volume associated with $m \times n$ matrices, *Lin. Algeb. and its Appl.* **167**(1992), 87-111